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Note

Note on the region of convergence of a polynomial series

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Abstract

The region of convergence of a polynomial series $\sum a_n q_n$ is determined, provided the (weak) asymptotic zero distribution of the sequence q_n and the nth root asymptotics of their leading coefficients is known.

Keywords: Polynomial series; Region of convergence; Logarithmic potential; Equilibrium distribution; Zero distribution

Let $\{q_n\}$ be a sequence of polynomials and $\{\alpha_n\}$ a sequence of complex numbers. The largest open set in which the series

$$\sum_{n=0}^{\infty} \alpha_n \, q_n(z) \tag{1}$$

converges locally uniformly is called the convergence region of series (1).

Peixuan [2] gives a partial characterization of possible regions of convergence of polynomial series as countable unions of pairwise disjoint simply connected domains. Moreover,

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under the assumptions that

$$\limsup_{n\to\infty} |\alpha_n|^{1/n} = \rho \in]0,1[\ , \qquad \sup_{|z|=1} |q_n(z)| = |q_n(1)| = 1$$

and that the zeros of the q_n are given by perturbed roots of unity $e^{i\theta_k^{(n)}}$ with

$$\frac{2(k-1)\pi}{n} < \theta_k^{(n)} \leqslant \frac{2k\pi}{n} \quad (k = 1, 2, \dots, n)$$

he shows that the region of convergence is the disk $\{z \in \mathbb{C} : |z| < 1/\rho\}$.

In this note we use potential theory to give a general description of the region of convergence when the asymptotic zero distribution in the weak-star sense and the nth root behaviour of the leading coefficients of the polynomials q_n are known. The results are formulated in terms of the following notions (see, for instance, [1,3]).

For a unit (Borel-) measure μ in $\mathbb{C}$, denote by

$$U^{\mu}(z) = \int \log \frac{1}{|z-\zeta|}\, d\mu(\zeta) \quad (z \in \mathbb{C})$$

its logarithmic potential. If q is a polynomial of degree k with zeros $\zeta_1, \dots, \zeta_k$ (taking into account multiple zeros), then v_q denotes the corresponding normalized zero counting measure, i.e., the unit measure $v_q = \frac{1}{k} \sum_{j=1}^{k} \delta_{\zeta_j}$ associating equal mass $1/k$ with each zero. Note that, if q is monic, then

$$U^{v_q}(z) = \log \frac{1}{|q(z)|^{1/k}} \quad (z \in \mathbb{C})\ . \tag{2}$$

For a compact set $E \subset \mathbb{C}$ there exists a unit measure μ_E on E with minimal energy

$$I(\mu_E) = \iint \frac{1}{|z-\zeta|}\, d\mu_E(z) d\mu_E(\zeta).$$

Moreover, by the Frostman theorem, $U^{\mu_E} \leqslant V_E := I(\mu_E)$ in $\mathbb{C}$ and $U^{\mu_E} = V_E$ quasi-everywhere (see below) on E. The measure μ_E is called equilibrium distribution of E, V_E is the Robin constant and $\operatorname{cap}(E) := e^{-V_E}$ the capacity of E. μ_E is unique if $\operatorname{cap}(E) > 0$. In this case, the non-negative function $g(z, \infty) := V_E - U^{\mu_E}(z)$ is referred to as the Green's function of $\mathbb{C} \setminus E$ with pole at ∞.

A relation is said to hold quasi-everywhere, if it holds everywhere, except for a set of zero capacity.

We say that a sequence $\{\mu_n\}$ of unit measures in $\mathbb{C}$ converges to a unit measure μ in the weak-star sense (notation: $\mu_n \xrightarrow{\star} \mu$), if

$$\lim_{n\to\infty} \int f\, d\mu_n = \int f\, d\mu$$

for every continuous function f with compact support.

In what follows, μ will be a unit measure with compact support. In addition, let $\{\alpha_n\}$ be an arbitrary sequence of complex numbers and set

$$\rho := \limsup_{n\to\infty} |\alpha_n|^{1/n}. \tag{3}$$

Theorem 1. *Suppose that* $\{p_n\}$ *is a sequence of monic polynomials of degree n with* $\nu_{p_n} \stackrel{\star}{\to} \mu$*. Assume that all zeros are contained in some fixed bounded set. Then the series*

$$\sum_{n=0}^{\infty} \alpha_n \, p_n \tag{4}$$

diverges quasi-everywhere in

$$\{z \in \mathbb{C} \, : \, U^{\mu}(z) < \log \, \rho\}.$$

Proof. Let $m \in \mathbb{N}$ be arbitrary and let

$$x \in K_m := \left\{ z \in \mathbb{C} \, : \, U^{\mu}(x) \leqslant \log \left(\rho - \frac{1}{m} \right) - \frac{1}{m} \right\}. \tag{5}$$

From (3) we deduce that there exists a subsequence $\Lambda = \Lambda(m) \subset \mathbb{N}$ such that

$$|\alpha_n|^{1/n} \geqslant \left(\rho - \frac{1}{m} \right) \quad (n \in \Lambda). \tag{6}$$

Now, suppose that series (4) converges in x . Then there exists n_0 such that $|\alpha_n| \, |p_n(x)| \leqslant 1$ for all $n \geqslant n_0$. By (6),

$$\begin{aligned} 1 \geqslant |\alpha_n|^{1/n} |p_n(x)|^{1/n} &\geqslant \left(\rho - \frac{1}{m} \right) |p_n(x)|^{1/n} \\ &= \left(\rho - \frac{1}{m} \right) \exp\{-U^{\nu_{p_n}}(x)\} \quad (n_0 \leqslant n \in \Lambda) \end{aligned}$$

or, taking into account (5),

$$U^{\nu_{p_n}}(x) \geqslant \log \left(\rho - \frac{1}{m} \right) \geqslant U^{\mu}(x) + \frac{1}{m} \quad (n_0 \leqslant n \in \Lambda). \tag{7}$$

On the other hand, the lower envelope theorem [3, Theorem I.6.9] and $\nu_{p_n} \stackrel{\star}{\to} \mu$ (along Λ) imply that

$$U^{\mu}(z) = \liminf_{\Lambda \ni n \to \infty} U^{\nu_{p_n}}(z) \tag{8}$$

quasi-everywhere, i.e., for all $z \in \mathbb{C} \setminus \Gamma$, where $\Gamma = \Gamma(m)$ is a set of zero capacity. By (7), $x \in \Gamma(m)$. The statement of the theorem follows since the countable union of sets of zero capacity is of zero capacity. □

Theorem 2. *Suppose that* $\{p_n\}$ *is a sequence of monic polynomials of degree n with* $\nu_{p_n} \stackrel{\star}{\to} \mu$ *and that all zeros are contained in some fixed bounded set. Then series* (4) *converges in*

$$\left\{ z \in \mathbb{C} \, : \, U^{\mu}(z) > \log \, \rho \right\}.$$

This convergence is locally uniform, provided U^{μ} *is continuous.*

Proof. Suppose first that U^μ is continuous. Let $K \subset \{U^\mu > \log \rho\}$ be compact, and let $\varepsilon > 0$ be such that $U^\mu \geqslant \log(\rho + \varepsilon) + \varepsilon$ on K. By the principle of descent [3, I.6.8],

$$\liminf_{n\to\infty} U^{\nu_{p_n}}(z) \geqslant U^\mu(z) \text{ uniformly on } K.$$

Hence, for some index $n_0 = n_0(\varepsilon)$,

$$\frac{1}{n} \log \frac{1}{|p_n(z)|} = U^{\nu_{p_n}}(z) \geqslant \log(\rho + \varepsilon) \quad (z \in K,\ n \geqslant n_0).$$

We may assume n_0 so large that, in addition, $|\alpha_n|^{1/n} \leqslant \rho + \varepsilon/2$ for all $n \geqslant n_0$. Therefore,

$$|\alpha_n|\,|p_n(z)| \leqslant (\rho + \varepsilon/2)^n\,(\rho + \varepsilon)^{-n} \quad (z \in K,\ n \geqslant n_0),$$

which implies

$$\sum_{k=n}^{\infty} |\alpha_k|\,|p_k(z)| \leqslant \sum_{k=n}^{\infty} \left(\frac{\rho + \varepsilon/2}{\rho + \varepsilon}\right)^n \quad (z \in K,\ n \geqslant n_0).$$

Thus, (4) converges uniformly in K.

If U^μ is not continuous, then the convergence in some point x can be derived similarly by considering $K = \{x\}$ in the previous arguments. □

In what follows, let $E \subset \mathbb{C}$ be a compact set with regular complement. Then $U^{\mu_E}(z) = V_E$ for all $z \in E$. $||\cdot||_E$ denotes the uniform norm on E.

Corollary. *Suppose $\{q_n\}$ is a sequence of polynomials of degree n with*

$$||q_n||_E = 1 \quad \text{and} \quad \nu_{q_n} \overset{\star}{\to} \mu_E.$$

Assume that all zeros are contained in a fixed bounded set. Then, for $\rho < 1$, the region of convergence of series (1) *is the set*

$$\left\{z \in \mathbb{C} : g(z,\infty) < \log \frac{1}{\rho}\right\}.$$

Example. Let $E = \{z \in \mathbb{C} : |z| = 1\}$, the constellation considered in [2]. Here, $g(z,\infty) = \log|z|$ for $|z| \geqslant 1$.

Proof of the Corollary. Write $q_n = \gamma_n\, p_n$, where p_n is monic. Then

$$1 = ||q_n||_E = |\gamma_n|\,||p_n||_E \geqslant |\gamma_n|\,||T_n||_E \geqslant |\gamma_n|\,\mathrm{cap}(E)^n,$$

where T_n is the nth Chebychev polynomial for E [3, Theorem III.3.1]. Thus,

$$|\gamma_n|^{1/n} \leqslant \mathrm{cap}(E)^{-1}. \tag{9}$$

On the other hand, for some $\zeta_n \in \partial E$,

$$1 = |q_n(\zeta_n)| = |\gamma_n|\,|p_n(\zeta_n)| \geqslant |\gamma_n|\,\exp\{-nU^{\nu_{p_n}}(\zeta_n)\},$$

or $U^{\nu_{p_n}}(\zeta_n) = \log|\gamma_n|^{1/n}$. Thus, by the principle of descent,

$$\liminf_{n\to\infty} \log|\gamma_n|^{1/n} \geqslant V_E = (U^{\mu_E})_{|E}.$$

With (9) it follows that the limit of $|\gamma_n|^{1/n}$ equals $1/\mathrm{cap}(E)$ and, therefore,

$$\limsup_{n\to\infty} |\alpha_n \gamma_n|^{1/n} = \frac{\rho}{\mathrm{cap}(E)}.$$

Theorem 2 yields that $\sum \alpha_n q_n = \sum \alpha_n \gamma_n p_n$ converges locally uniformly in

$$\{z : U^{\mu_E}(z) > \log\rho - \log\mathrm{cap}(E)\} = \{z : g(z,\infty) < \log 1/\rho\}. \tag{10}$$

Since $\rho < 1$, any not-empty open set which is larger than the region (10) intersects with $\{z : U^{\mu_E}(z) < \log\rho - \log\mathrm{cap}(E)\}$. Hence, by Theorem 1, (10) is in fact the region of convergence of the series under consideration. □

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